JOURNAL OF AIRCRAFT
Vol. 34, No. 5, September–October 1997

Extended-Betz Methods for Roll-Up of Vortex Sheets

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The method derived for the roll-up of vortex sheets shed by lifting wings has been extended so that it can be applied to complex wakes that roll up into more than one vortex pair. Although these so-called extended-Betz methods often predict realistic vortex structures that approximate measured ones, the guidelines need to be improved if the predictions are to be reliable enough for such purposes as the assessment of the hazard posed by a wake or for the validation of other theoretical methods. This paper first describes the problem and then considers whether the five vortex invariants for the time-dependent motion of two-dimensionlal vortices can provide improved guidelines. In particular, the vortex invariants are studied to find out if they can be used to determine the initial division of the vortex sheet for separation into multiple vortices, and then possible merger into larger vortices. It is concluded that any refinements of the present forms of the extended-Betz methods to better define the structure of multiple vortex wakes will be very difficult and probably cannot be derived from the vortex invariants.

Nomenclature

b	= wing span, ft
C_L	= lift coefficient, L/qs
c	= wing chord, ft
L	= lift, lb
N	= total number of vortices
q	= dynamic pressure, $\rho U_\infty^2/2$, lb/ft^2
r	= radius, ft
S	= wing planform area, ft^2
T	= kinetic energy of flowfield, slug/ft s^2
t	= time, s
U_∞	= freestream velocity, ft/s
u, v, w	= velocity components in x, y, z directions, ft/s
W	= Kirchhoff–Routh path function
x	= distance in flight direction, ft
y	= distance in spanwise direction, ft
z	= distance in vertical direction, ft
Γ	= circulation bound in wing, ft^2/s
γ	= circulation in point vortex, ft^2/s
ρ	= air density, slug/ft s^2

Subscripts

i, j	= indices
n	= vortex number
s	= plane of vortex sheet
v	= plane of rolled-up vortex
w	= wing plane
0	= centerline value

Introduction

THE hazard posed by the vortex wakes of subsonic transport aircraft is currently being circumvented by specifying that the in-trail distances between aircraft during landing and takeoff operations are large enough so that hazardous wakes are not encountered. Unfortunately, the spacings used are larger than required for other air traffic factors, so that the full capacity of airport runways is not realized. In an effort to increase the capacity of airports, research efforts have been directed at developing methods that will predict the structure of vortex wakes to better evaluate the hazard, and finding those characteristics of the wake-generating wings that make their vortices less hazardous.

In support of the goal for greater airport capacity, this study is an attempt to improve the present capability to predict vortex structure from the span loading on the generating wing. In particular, the present study examines the simple method introduced by Betz,[1] and its extensions, which relate the span loading on a wing (or the vortex sheet that it sheds) to the rolled-up vortex far downstream (Fig. 1). The derivation of the basic Betz method[1] is based on three of the five available invariants for the time-dependent motion of two-dimensional vortices. To produce a unique result, it was also necessary for Betz[1] to specify the order in which vorticity or circulation is rolled into or layered onto the vortex center. Based on experimental observation, it is specified that the vortex sheet rolls up sequentially from the wingtip inboard to the wing centerline. The Betz roll-up theory came into active use during the 1970s, when research into the hazard posed by vortex wakes of subsonic transport aircraft increased substantially because of safety and capacity concerns at airports.[2–12] As a consequence of the increased use of the Betz method,[1] several researchers noted a simplified version of the roll-up equations at about the same time.[4–6] The simplified roll-up relationships presented in Fig. 1 are derived by means of the integral forms of three vortex invariants for two-dimensional vortices.[4] To arrive at a viable method, Betz[1] assumed that the vortex invariants could be applied to portions of the vortex system rather than to the entire flowfield. The method and sequence of application chosen by Betz[1] are based on experiment and on a knowledge of fluid mechanics, so that the results are surprisingly accurate. The small cross-hatched areas in Fig. 1 trace how a vortex element proceeds from the vortex sheet at the trailing edge of the wing, through the roll-up process, to fill an annulus of the fully developed vortex far behind the wing.

The Betz method[1] has been found to predict the vortex structure when the assumptions in the method are valid. Usually, the primary requirement of the span loading on the wake-generating wing is that the vortex sheet it sheds should decrease in strength monotonically from the wingtip to the centerline, so that a single vortex pair is produced in the wake.[6,7] Because it is assumed that the rolled-up vortex is axially symmetric, it is also required that any other vortices in the wake be far enough away so that the streamlines can be approximated by

Received Oct. 2, 1996; revision received Feb. 26, 1997; accepted for publication May 24, 1997.

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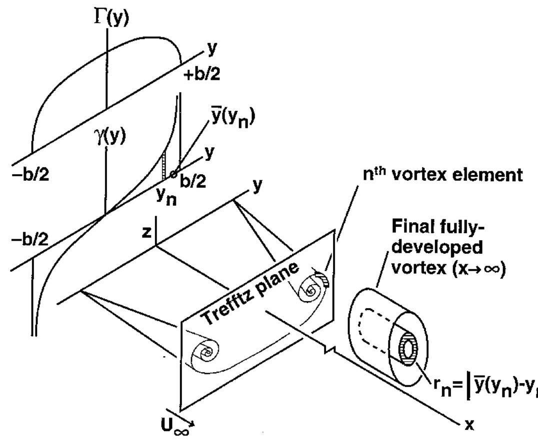


Fig. 1 Diagram[4] of the Betz method[1] for relationship among span loading on wing, vortex sheet, and final rolled-up vortex.

concentric circles. If, however, the span loading on the generating wing is such that multiple vortex pairs persist into the wake far behind the wing, the formulation by Betz[1] is inadequate; e.g., for triangular span loading and for certain span loadings carried by wings with flaps for landing and takeoff.[4−12] Because the basic method is simple, researchers were motivated to stay with the basic Betz analysis[1] and simply extend its capability by the addition of guidelines to eliminate the unrealistic results produced by the Betz method[1] when the span loadings and vortex sheet strengths are not monotonic.[4,6] These so-called extended-Betz methods[1] use guidelines based more on intuitive rules rather than on an approach that has a theoretical basis. It is found that the extended guidelines do eliminate certain undesirable features of the original Betz theory[1] (such as multiple-valued swirl velocities), and also provide a rational basis for analyzing more complex vortex wakes.[8−11] However, close scrutiny of the guidelines that determine the circulation content (and its distribution) within each of the vortices, indicates that the roll-up procedures are inadequate for the general case when experimental accuracy is required.[8,12] Because the guidelines added to the Betz method[1] to date do not consistently provide reliable results, they either need to be improved upon or exchanged for a new analysis procedure. The study reported here indicates that it is necessary to go to a new analysis procedure.

The objective of this study is to find improved guidelines for the rolled-up structure of wakes composed of multiple vortex pairs. After the problem has been described and illustrated, possible sources for improved guidelines are explored. In particular, the values of the vortex invariants are examined and then compared with each other during the roll-up process to determine whether they give any indication as to how a complex vortex sheet should be divided and rolled up to form a wake composed of multiple vortices. To facilitate the analysis, this study begins by writing the vortex invariants for a system of two-dimensional point vortices in an inviscid and incompressible fluid.[13,14] The first three invariants are then used to rederive another form of the Betz formulation.[1] Because only three of the invariants are used in Betz' analysis,[1] a determination is next made as to whether the two unused vortex invariants are conserved as the vortices in the vortex sheet are incorporated into the rolled-up vortex. Because all of the invariants are found to be conserved throughout the one-sided roll-up process, an energy formulation is derived and then compared with the formulation based on the second moment of circulation. The variation of the vortex invariants as a function of the spanwise roll-up sequence is then studied for several different span loadings. The objective of this exercise is to find out if any of the invariants provide an indication of the proper division of the vortex sheet into segments for multiple vortex wakes, or for the proper wrapping sequence of the vorticity in a given vortex. It is concluded that the vortex invariants do not by themselves provide the guidelines being sought, and that the necessary criteria or equations must be found elsewhere; e.g., from experiment or from numerical analysis. It is also concluded from the numerical analysis of several vortex sheets that the sheet division and roll-up process can be very complicated and prolonged, so that the final disposition of vortices may be quite different from that anticipated when a prediction is based only on the structure of the vortex sheet at the trailing edge of the wing. It is also concluded that improvements in the Betz procedures,[1] that would include items such as the effect of the opposite side of the wake and the motion of the entire vortex system downward, require integrations of the invariant relationships that cannot be carried out analytically in simple terms and are therefore not practical. The analysis presented here is restricted to two-dimensional, inviscid, and incompressible flow and does not include variations that might occur in the vortex wakes of helicopters.[15] Furthermore, variations in the streamwise velocity as a function of radius in the vortex will also be assumed to have a negligible effect on vortex structure.[3,9,15,16]

Background

The extensions applied[4,6] to the basic formulation of Betz[1] treat two aspects of the roll-up procedure that were not considered in the original analysis. The first relates to the division of vortex sheets for roll-up into multiple vortices on each side of the centerline. Once the vortex sheet has been properly divided, the second aspect concerns how, and in what sequence, the vorticity in the vortex sheet should be layered around the center of each rolled-up vortex in the wake. Guidelines presented in Refs. 4 and 6 recommend that the vortex sheet be divided for separate vortices at locations where the vortex sheet ends, passes through zero (or has minimum strength), and/or where the self-induced vertical velocity passes through zero. The centers about which each vortex begins its wrapping sequence of vorticity are chosen at those locations where the vortex sheet ends, and/or where the sheet has a maximum strength. When the vortex center is located at the ends of a sheet segment, it forms a single spiral as it progresses from its originally near-flat structure to the final axially−symmetric fully developed vortex. When a vortex center is located somewhere in the middle of a vortex sheet segment, and not at an end, vorticity is incorporated into the rolled-up vortex from both sides of the chosen center. That is, the sheet rolls up as a double spiral from a location within the vortex sheet segment. Examples of the procedure when applied to idealized span loadings illustrate how some vortex structures generated by the original Betz method[1] are changed into realistic vortex structures[4] and are found to compare favorably with experiment.[8] The procedures employed by Ref. 6 organized and interpreted flight measurements of vortex structures under difficult conditions. Recent application[12] of the extended-Betz method of Ref. 6 produced good agreement with experiment in several examples, but in some other cases, the agreement was not as good. As noted previously, it was concluded in Ref. 12 that extended-Betz methods can produce acceptable results in a number of cases, but that the guidelines need to be improved. As mentioned in the Introduction, the study reported here indicates that it is necessary to go to a new analysis procedure.

Before discussing the Betz methods,[1] several simple examples are presented to indicate the complications that can arise when the vortex wakes of aircraft are analyzed. The examples presented in Figs. 2–4 for three different span loadings were chosen to illustrate the effect of the structure of the span loading, or vortex sheet strength, on sheet division and roll-up structure of the final vortex wake. The dimensionless label with each of the plots for the three cases correspond roughly to 0, 5, and 10 span lengths behind the wake-generating wing. The axes have been shifted downward in all cases so that they

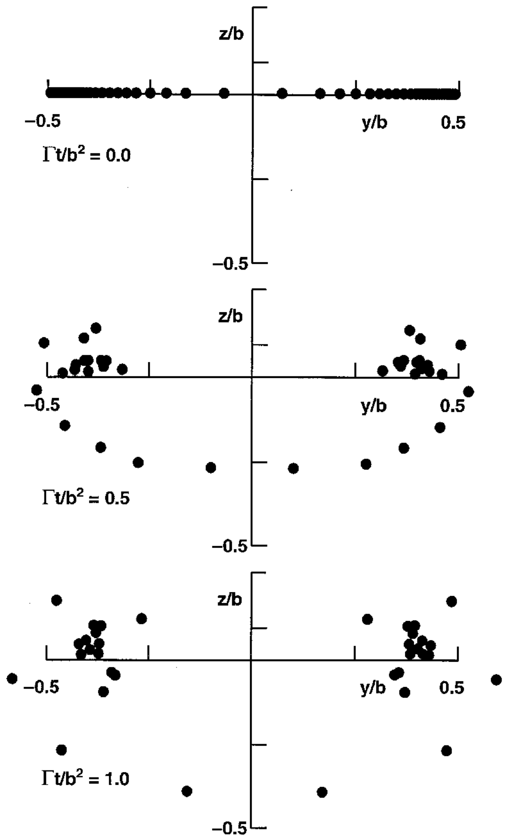


Fig. 2 Time-dependent roll-up of vortex sheet shed by elliptic loading; N = 26.

remain near the vortex locations. Elliptic loading $\{\Gamma(y)/\Gamma_0 = [1 - (2y/b)^2]^{1/2}\}$ is chosen first to demonstrate how the vortex sheet rolls up from the wingtip inboard without sheet division (Fig. 2). In contrast, a vortex sheet of constant strength [as shed by an isolated triangularly loaded wingtip; $\Gamma(y)/\Gamma_0 = 1 - |(2y/b|$ rolls up from each end so that the vortex sheet is divided in half to form two vortices of equal strength and structure (Fig. 3). (The port side of the wake is left blank in Fig. 3 to emphasize the one-sided character of the example.) In Fig. 3, the four-point vortices in the center of the vortex sheet are represented by open rather than filled symbols. From the movement of these four vortices, it becomes clear that, as expected, a single vortex sheet of constant strength divides at its center to form two equal vortices. If, however, the vortex sheet is shed by an entire wing that is triangularly loaded, the inboard ends of the vortex sheets from the port and starboard sides of the wake are adjacent and opposite in sign. The two sheets of constant and opposite strength now influence the division process so that the two vortices on each side of the centerline are not of equal magnitude or structure (Fig. 4). Rather, the influence of the vorticity on the opposite side of the centerline causes the division point of the two vortex sheets to be offset so that all four of the point vortices at the center of each sheet are incorporated into the outboard vortices, to form two vortex pairs of unequal strength. Another guideline suggests that sheet division be chosen at those spanwise locations where the up- or downwash velocities vanish. Such a criterion is also inadequate because such a guideline indicates that three rather than four-point vortices join the outboard vortex, which is not what actually occurs.

The examples in Figs. 2–4 have a far simpler roll-up process than the vortex sheets shed by wings with flaps deployed and with wing-mounted engines. If the drag of the wing, its high-lift elements, and the thrust of the engines are added to the self-induced convection of the circulation in the wake, the

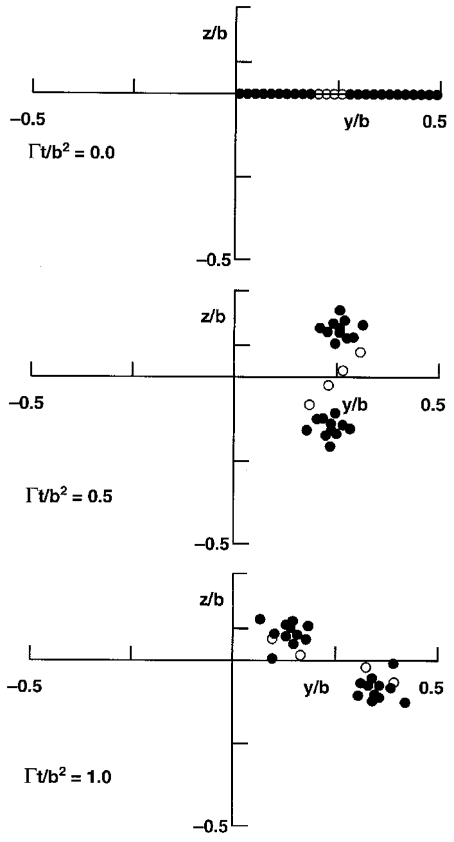


Fig. 3 Time-dependent roll-up of vortex sheet shed by isolated wingtip with triangular loading, i.e., vortex sheet of constant strength; N = 26.

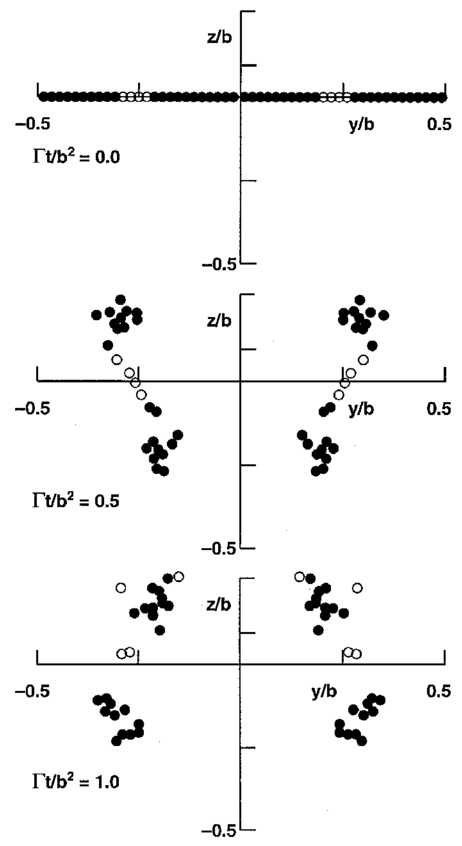


Fig. 4 Time-dependent roll-up of vortex sheet shed by triangular loading, i.e., two adjacent vortex sheets of constant and opposite strength; N = 26.

formation of the final configuration of vortices is even more complicated, thereby requiring more extensions than discussed here. It is also important to note that the use of point-vortex calculations to simulate the dynamics of vortex wakes is simple, convenient, and representative of a number of characteristics of vortex wakes. However, the method does have faults. Firstly, the accumulation of vorticity or circulation into discreet locations prevents simulation of the stretching and spreading of the circulation in a vortex sheet over the spiral shape and/or annulus during the roll-up process. It also does not include the slow asymptotic assimilation of the remote weak segments of the sheet into the final vortex structure. The point-vortex representation also sometimes yields unrealistic motions when two or more vortices approach one another closely because the velocity is singular at the center of a point vortex.

Invariants for Motion of Vortices

The invariants for the time-dependent motion of two-dimensional vortex systems[13,14] relate a given function or state at a given time, i.e., at the wing trailing edge, to the state of the system at another time, i.e., at a station far behind the wing. To facilitate the analysis to follow, the invariants presented in various texts[13,14] are now written for vortex sheets that are approximated by point vortices whose strengths are given by γ_i and locations by (y_i, z_i).

Conservation of circulation:

$$\Gamma_s = \Gamma_w = \sum_{i=1}^{N} \gamma_i = \text{const} \tag{1}$$

Conservation of first moment of circulation:

$$\bar{y}_N \Gamma_s = \sum_{i=1}^{N} \gamma_i y_i = \text{const} \tag{2a}$$

$$\bar{z}_N \Gamma_s = \sum_{i=1}^{N} \gamma_i z_i = \text{const} \tag{2b}$$

Conservation of second moment of circulation:

$$J_s = \sum_{i=1}^{N} \gamma_i[(y_i - \bar{y}_N)^2 + (z_i - \bar{z}_N)^2] = \text{const} \tag{3}$$

Conservation of angular moment of circulation:

$$M_s = \sum_{i=1}^{N} \gamma_i[(y_i - \bar{y}_N)w_i - (z_i - \bar{z}_N)v_i] = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{\gamma_i \gamma_j}{2\pi} = \text{const} \tag{4}$$

Conservation of energy:

$$W_s = -\frac{\rho}{4\pi} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \gamma_i \gamma_j \, \ell n[(y_i - y_j)^2 + (z_i - z_j)^2] = \text{const} \tag{5}$$

where N is used to represent the vortex sheet. It is emphasized here that the vortex invariants strictly apply only when all of the circulation in the flowfield is included in the various summations. If such a requirement is strictly enforced and the vortex invariants are applied to the vortex system as a whole, very little information is obtained because Eqs. (1) and (3) vanish and the five summations usually contain a much larger number of unknowns. Therefore, the Betz method[1] achieves a solution for the roll-up problem by specifying that the invariants be applied progressively to portions of the flowfield so that the number of unknowns does not exceed the number of equations available for a solution. Results achieved by the Betz method[1] are in good agreement with experiment, even though the invariants are applied to only certain portions of the vortex system at a time, because the assumptions made and the guidelines given by Betz[1] are valid for the way that the invariants are used.

When the vortex invariants are applied to portions of the vortex system, in Eq. (1) the portion of the vortex sheet that has been incorporated into the vortex by the Betz method[1] is also equal to the circulation bound in the wing at that spanwise station, i.e., $\Gamma_w(y_n) = \Gamma_s(y_n)$, so that the two can be used interchangeably. Equations (3) and (4) are written so that the centroid of circulation $(\bar{y}_N, \bar{z}_N)$ is used as the reference location on which the second moment and angular moment are computed to put the equations in the forms needed for the roll-up analysis. The equations are still applicable even if another reference location is used. The function W in Eq. (5), which is often referred to as the Kirchhoff–Routh path function, is sometimes written with a positive sign. The negative sign is correct, however, because it represents the variable portion of the kinetic–energy content (usually designated by T) of the flowfield.[14] When the integration over the flowfield is carried out to determine the energy, the positive part is found to be a constant. The positive part is also found to become infinitely large when the area of integration is expanded to include the core region of the point vortices and the region at very large radius.[13,14] Because the infinite parts are constant with time, they are usually left out of any analysis of time-dependent vortex motions.[14]

In Eqs. (1–5), and those to follow, the subscript s is used to denote quantities in the across-stream (or Trefftz) plane located at the trailing edge of the wake-generating wing where the vortex sheet begins. The subscript v is used to denote quantities at the Trefftz plane located far downstream from the wing where the vortices in the wake are assumed to be fully rolled up (Fig. 1). The subscript w is used to refer to the wing itself and to quantities associated with the span loading. The spanwise and vertical velocity components in Eq. (4) for the ith vortex are given by

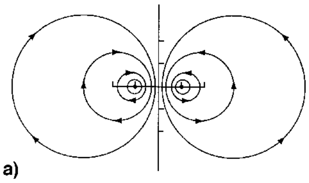
$$v_{si} = -\frac{1}{2\pi} \sum_{j=1}^{N} [\gamma_i(z_i - z_j)]/[(y_i - y_j)^2 + (z_i - z_j)^2] \tag{6a}$$

$$w_{si} = +\frac{1}{2\pi} \sum_{j=1}^{N} [\gamma_i(y_i - y_j)]/[(y_i - y_j)^2 + (z_i - z_j)^2] \tag{6b}$$

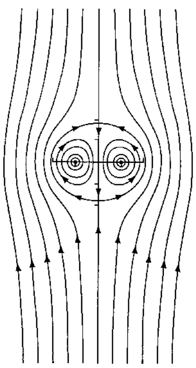
Roll-Up Procedure Based on Second Moment

Assumptions

The basic- and extended-Betz methods[1] both consider only two states of the vortex system. The first state consists of the essentially flat vortex sheet as it is shed at the trailing edge of a lifting wing (Fig. 1). The second state consists of the rolled-up or fully developed axially symmetric vortices in a Trefftz plane located far behind the wake-generating wing. The assumption of axial symmetry is approximately valid when the portion of the wing wake being analyzed is far enough in the spanwise direction from the opposite vortex so that the final Trefftz-plane streamlines do not deviate far from circular. In both methods, vortex invariants are used to relate the radius at which a portion of the circulation in the vortex sheet at the trailing edge of the wing is deposited in the fully developed vortex. (The strength of the vortex sheet is, of course, proportional to the derivative of the span loading on the wing.) Betz' analysis[1] does not concern itself with the time-dependent motion of the vorticity from one state to the other. In the strictest sense, the invariants require that all of the vortices or vorticity in the flowfield be included in its evaluation. As indicated here, the Betz method[1] and the analysis to follow, apply the invariants to only certain portions of the vortex system at a time. A justification for such an assumption at the station far downstream from the wing is that the swirl velocity in an axially



a)



b)

Fig. 5 Streamlines for the flowfield of two equal and opposite point vortices designated by filled symbols: a) vortices forcibly held in place, and b) vortices moving under self-induced velocity field.

symmetric vortex depends only on the circulation contained within the radius where the velocity is being determined, and is not affected by circulation outside of that radius that is concentric with the inner circular streamlines.

The simplicity of the basic-Betz method[1] results from several assumptions. First, it is assumed that the span loading at the wingtip being analyzed is far from the vorticity being shed by the opposite side of the wing. This assumption permits the streamlines in the rolled-up vortex to be approximated by concentric circles so that the analysis is greatly simplified. If the circulation and rolled-up vortex on the opposite side were included in the analysis, the streamlines would be offset laterally as illustrated in Fig. 5a for two-point vortices. Similarly, the addition of a second equal and opposite vortex to the flowfield causes the vortex pair to move downward because of the mutual induction of the velocity field of the two vortices. If this downward motion is included in the analysis, the streamlines associated with the vortex pair and the circulation distribution are confined to a vortex oval[13] (Fig. 5b). In Fig. 5, the spanwise distance between the vortices has been kept the same to illustrate the differences in the streamline locations and shapes when the vortices are allowed to move. As a consequence, the influence of the opposite vortex causes the streamlines to be enclosed within, and to be convected within, an oval. The streamlines are then no longer circles and the approximate centers of rotation are offset laterally (Fig. 5). Furthermore, the location of the circulation associated with each streamline is quite different in the two cases shown in Fig. 5, which are also different from the one assumed by Betz.[1] These changes should be incorporated into any upgraded or extended method, but they increase the complexity of the roll-up analysis to the point where it becomes necessary to resort to numerical methods for a solution.

Betz' original method[1] next assumes that the vortex sheet shed by the wing rolls up in an orderly fashion from the wingtip inboard, so that successive layers of the vortex sheet are wrapped around the center and previous wrappings (Fig. 1). It is found from the study of experimental and idealized hypothetical span loadings that the vortex sheet shed by a wing does roll up from the wingtip inboard in an orderly fashion if the span loading and the vortex sheet that it sheds increases monotonically from the wingtip inboard.[2–12] Of these two criteria, the second overrides and includes the first, so that it is only necessary to specify that the strength of the vortex sheet increases monotonically from the wingtip inboard to the wing centerline. It is difficult to derive a precise dividing line between those span loadings that produce an orderly roll-up as assumed by Betz[1] and those that shed vortex sheets that roll up about vortex centers other than the wingtip. Vortex sheets shed by other span loadings that are more typical of aircraft in their landing configurations, have multiple vortex centers. Analysis of these span loadings can be carried out by the use of extended-Betz methods[1] that require more assumptions regarding the roll-up process. As mentioned previously, it has been found that further refinements of both the original-Betz and the extended-Betz methods[1] require inputs from other theoretical and/or numerical methods as to the roll-up process that greatly complicates the methods and may not increase its reliability.

Finally, the conservation of circulation between the two states is combined with the second moment of the circulation in the two planes to determine the radius in the rolled-up vortex where the circulation from the vortex sheet shed by the wing is deposited. As mentioned previously, the vortex invariants apply strictly only when all of the vortices in the flowfield are included. When the Betz method[1] assumes that the opposite wingtip and its shed vorticity is remote, application of the second moment of vorticity to only one side is approximate. Because the influence of the opposite vortex fades as the square of the lateral distance between the vortex centers, the influence is present throughout the roll up and is probably only negligible near the centers of the vortex where high rotational velocities dominate.

Roll-Up Equations

In this section, the basic-Betz roll-up method[1] is derived by use of the first three invariants written in their point-vortex form; i.e., Eqs. (1–3). As with the derivation of the integral form of the roll-up equations, the center of the rolled-up vortex is located at the centroid of circulation $(\bar{y}_n, \bar{z}_n)$. The conservation of circulation as it is transferred from the vortex sheet behind the wing into the fully developed vortex far behind the wing is now written as

$$\Gamma_v(r_n) = \Gamma_s(y_n) = \sum_{i=1}^{n} \gamma_i \tag{7}$$

where it is assumed that the point vortices are numbered beginning with the point vortex nearest the wingtip and then proceed inwardly to the Nth vortex that is nearest the centerline of the span loading. In Eq. (7), the subscript n is used to designate the number of vortices that have been rolled up into the final vortex structure. The radius r_n that encloses the circulation in the vortex is related to the corresponding spanwise station on the wing y_n by the second moment of circulation written for the station at the wing as

$$J_{sn} = \sum_{i=1}^{n} \gamma_i[(y_i - \bar{y}_n)^2 + (z_i - \bar{z}_n)^2] \tag{8a}$$

and at the rolled-up vortex as

$$J_{vn} = \sum_{i=1}^{n} \gamma_i r_i^2 \tag{8b}$$

At each stage of the roll-up, the second moment of circulation is conserved as each segment of vortex sheet is transferred to the rolled-up vortex that requires that $J_{vn} = J_{sn}$.

Similar equations are now written for the $(n - 1)$th point

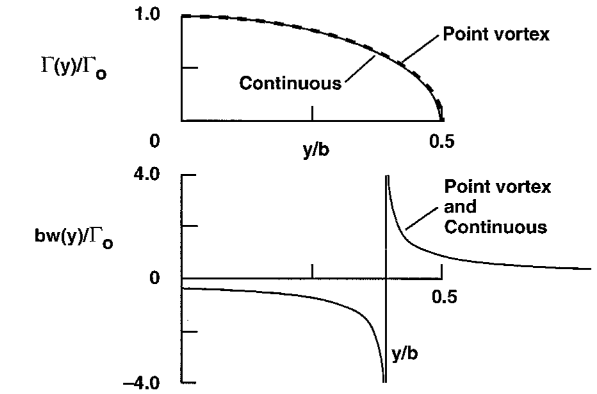


Fig. 6 Comparison of span loading and up- and downwash for rolled-up vortex structure as predicted by continuous and point-vortex methods using second moment of circulation; uniform spacing, N = 51; ——, continuous; - - - - -, point vortex.

vortex. The difference between the equations written for the (n)th and the $(n - 1)$th point vortex systems then becomes

$$\Delta J_{vn} = J_{vn} - J_{v(n-1)} = \gamma_n r_n^2 \tag{9a}$$

$$\Delta J_{sn} = J_{sn} - J_{s(n-1)} \tag{9b}$$

Therefore, when increments in the second moment of circulation for the two planes are equated $\Delta J_{vn} = \Delta J_{sn}$, the radius at which the circulation γ_n from the nth vortex is deposited is given by

$$r_n = \{(1/\gamma_n)[J_{sn} - J_{s(n-1)}]\}^{1/2} \tag{10}$$

where the circulation contained in the nth vortex is spread over an annulus or ring of zero thickness at the radius r_n.

The difference between the second moments for the vortices in the wing plane, J_{sn} and $J_{s(n-1)}$, cannot be combined for a simplification because the centroids of the vortices for the nth and the $(n - 1)$th vortex systems are usually not exactly the same. As the spanwise increments in the span loading become smaller, the difference between the locations of the centroids for successive point vortices also becomes smaller so that, in the limit of a large number of vortices, they are equal. The roll-up equations for the point vortex and integral formulations are then the same. In a numerical example, the main difference between the results predicted by the integral and point-vortex formulations is brought about by the fact that the first point vortex has its entire circulation located at the center of the rolled-up vortex. In the integral formulation, the circulation from the first vortex is spread over a disk whose radius is determined by the equations in Fig. 1. This difference near $r = 0$ causes an offset in the first and all other radii in the distribution of circulation in the rolled-up vortex. If the radial offset of the first vortex is added to the results for the point-vortex formulation, the two results are brought into agreement. With this adjustment, the difference between the vortex structure predicted by the two formulations is negligible when at least 50 subdivisions in the span loading are used in the roll-up procedure.

To assemble the equations needed to compare the results of the two formulations, the equations derived when the span loading is treated as continuous and the invariants are written in their integral form are repeated here. From Ref. 4, the relationship between r_n and y_n that corresponds to Eq. (10) is

$$r_n = |\bar{y}(y_n) - y_n| \tag{11}$$

Equation (11) may also be written as

$$r_n = -\frac{1}{\Gamma_w(y_n)} \int_{b/2}^{y_n} \Gamma_w(y)\, dy \tag{12}$$

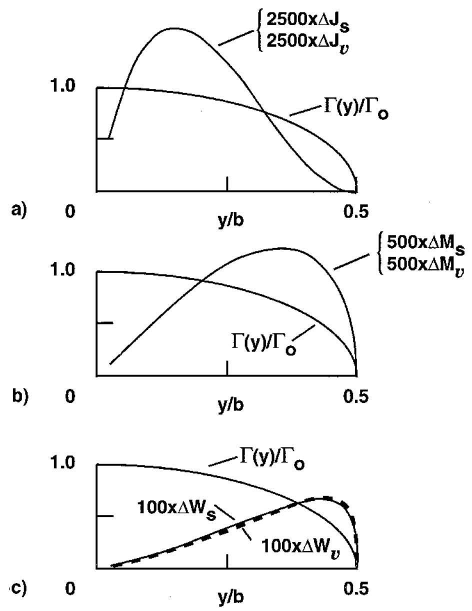


Fig. 7 Comparison of incremental values of vortex invariants for elements in vortex sheet and in rolled-up vortex as predicted by second moment of circulation; cosine spacing, N = 51; —— vortex sheet; - - - - - -, rolled-up vortex: a) second moment, b) angular moment, and c) energy.

With the roll-up relationships available for both the continuous and point-vortex representations of the vortex sheet, the two methods are compared in Fig. 6 for elliptic loading. It is found that the up- and downwash velocity distributions predicted by Eqs. (10) and (11) are the same for all practical purposes. In Fig. 6, the two span loadings appear to differ slightly. The difference arises because the spanwise location of the point vortex is at the center of the segment of vortex sheet that it represents, which tends to elevate the value for the bound circulation a small amount over that for the continuous span loading. It is also of interest to plot the magnitude of the increment in the second moment of circulation at each step of the incorporation of a point vortex from the vortex sheet into the rolled-up vortex (Fig. 7a).

Comparison of Invariants During Roll-Up

In the Betz method,[1] the incremental values of the three vortex invariants used in the formulation do not change as the piecewise elements are taken from the vortex sheet and incorporated into the rolled-up vortex. In this section, the increments for the second moment of circulation are compared with the invariants for angular moment and for the energy to find out if, during the transfer, the equality holds not only for the second moment, but for the other invariants as well.

Invariant for Angular Moment

The invariant for the angular moment of the circulation about its centroid, Eq. (4), is not used in Betz' derivation of the equations for vortex structure. A determination is now made as to whether the invariant's value is sustained from the wing to the vortex plane as the roll-up procedure progresses from the vortex center to its outer limit. Because the invariant for the angular moment of the circulation is presumably an independent invariant for vortex motion, this may not necessarily be true. It was found, however, that the numerical values of the invariant for the vortex sheet and for the rolled-up vortex planes at each step in the roll-up sequence are exactly the same (Fig. 7b). Because such an outcome seemed unusual, Eqs. (2) for the centroids and Eqs. (6) for the velocity components in the wing plane were inserted into the equation for M_s given by Eq. (4). It was found that the resulting dependence of M_s on the locations of the vortices disappeared and the value reduces to the double summation on the right of Eq. (4). The invariant for the angular moment of the circulation appears then to be a check on the computational accuracy of the velocity components and not necessarily a completely independent equation. Equation (4) was then found to not be useful for the development of a roll-up procedure nor for a determination for occurrences when the roll-up sequence is proceeding incorrectly.

Invariant for Energy

A determination is now made as to whether the invariant for the energy is conserved during the roll-up process as assumed by the Betz methods. As stated previously, the invariant for energy is not used in the derivation and, therefore, does not necessarily need to be conserved at the various piecewise stages of the roll-up of the vortex sheet. To follow the invariant for the energy during the incorporation of vortices into the final vortex, the increment in energy for a vortex element from the vortex sheet is written for the nth vortex element as

$$\Delta W_{sn} = W_{sn} - W_{sn-1} \tag{13}$$

The invariant for the energy of the vortex sheet is, from Eq. (5), written for the roll-up process as

$$W_{sn} = -\frac{\rho}{4\pi}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\gamma_i\gamma_j\,\ell n[(y_i - y_j)^2 + (z_i - z_j)^2] = \text{const}$$

where n is the number of vortices being incorporated into the rolled-up vortex, beginning as before with the vortex located at the wingtip, which is usually the strongest.

Because an equation for the energy of a system of concentric vortex rings in the rolled-up vortex plane is not available, it is derived by integration of the equation for the kinetic energy of the fluid between the rings. Because the fully developed vortex is assumed to be axially symmetric, the energy T_v is given by

$$T_v = \frac{\rho}{2}\int_0^\infty v_\theta^2 2\pi r\,\mathrm{d}r \tag{14}$$

where the swirl velocity for $(r_{n-1} < r < r_n)$ is given by

$$v_{\theta n} = \sum_{i=1}^{n-1}\frac{\gamma_i}{2\pi r} \tag{15}$$

The integration for the energy in the flowfield goes from the radius of one vortex ring to the next. Because the circulation from a given point vortex is spread over an annulus of zero thickness, no energy is contributed by the ring itself. Furthermore, it appears that no energy is expended by the process that distributes the circulation contained in the point vortex around the annulus. Because the first point vortex is deposited at the center of the rolled-up vortex at zero radius, the first integral in Eq. (14) is given by

$$T_{vn} = \frac{\rho}{2}\int_0^{r_2}\frac{\gamma_1^2}{2\pi r}\,\mathrm{d}r = \frac{\rho}{4\pi}\gamma_1^2[\ell n(r_2) - \ell n(r)_{r\to 0}] \tag{16}$$

As pointed out by Batchelor,[14] the contribution to the energy by the term that goes to infinity as $r \to 0$ is constant with time and is therefore ignored. Similarly, the contribution to the energy by the fluid outside of the radius of the outermost ring also goes to infinity at the upper limit of integration. Once again, it is constant with time and is ignored. The symbol W_s (rather than in T_s) is used to denote the time-varying portion of the kinetic energy in the flowfield of the vortex sheet because it does not contain the infinite contributions to the invariant for energy. Similarly, the parameter W_v is used to denote the finite and time-varying quantities for the energy in the flowfield of the rolled-up vortex. The finite and variable parts of the energy of the rolled-up vortex where the point vortices are deposited are then

$$W_{vn} = \frac{\rho}{4\pi}\left[\gamma_1^2\,\ell n(r_2) + (\gamma_1+\gamma_2)^2\,\ell n\left(\frac{r_3}{r_2}\right) + (\gamma_1+\gamma_2 + \gamma_3)^2\,\ell n\left(\frac{r_4}{r_3}\right) + \cdots + \left(\sum_{i=1}^{n-1}\gamma_i\right)^2\ell n\left(\frac{r_n}{r_{n-1}}\right) - \left(\sum_{i=1}^{n}\gamma_i\right)^2\ell n(r_n)\right] \tag{17}$$

The difference in the energy between the nth and the $(n-1)$th step is found by subtraction to be

$$\Delta W_{vn} = \frac{\rho}{4\pi}\left[\left(\sum_{i=1}^{n-1}\gamma_i\right)^2 - \left(\sum_{i=1}^{n}\gamma_i\right)^2\right]\ell n(r_n) = \Delta W_{sn} \tag{18}$$

A comparison is made in Fig. 7c of the energy increments in the wing plane with the energy increments in the rolled-up plane when the roll-up radii are based on the second moment of circulation. The results in Fig. 7c indicate that the energy increments are close, but not in perfect agreement. Because point-vortex formulations are used to derive the energy in both planes, it appears that the energy is approximately, but not exactly, conserved when the second moment of circulation is used in a piecewise process to carry out the roll-up procedure. No extreme variations between the two results were observed in the cases tested, which suggests that large errors in energy levels do not occur. The agreement at the end of the vortex sheet where the circulation goes to the center of the rolled-up vortex is usually not as good as the rest of the vortex sheet. The generally good agreement observed in Fig. 7c persists when computations were carried out for other span loadings. It was found that the shape of the curves did change as the span loadings changed, and also as the vortex spacings changed, i.e., uniform or cosine distributions, but the conservation of energy was just as good in all of the cases tested. It was also noted that when cases were computed, wherein the vortex sheet should have rolled up into two vortices rather than just one, e.g., triangular loading, the energy relationship did not provide any kind of signal to indicate that the progressive roll up of a vortex sheet about one vortex center, rather than two, was improper.

Roll-Up Procedure Based on Invariant for Energy

The difference in energy (rather than second moment) between the nth and the $(n-1)$th vortex element [Eq. (18)], is now used to calculate the radius r_n at which the circulation is deposited in the rolled-up vortex plane. The comparison in Fig. 8 of the up- and downwash distribution calculated by use of the invariant for energy [Eq. (18)], with that calculated by means of the second moment, indicates that both yield about the same structure. When applied to elliptic and triangular span

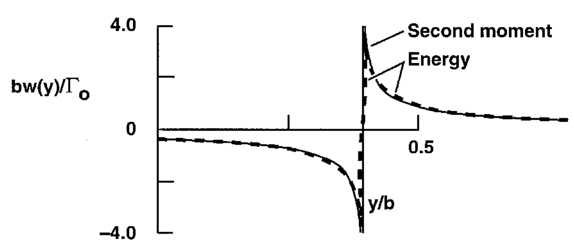


Fig. 8 Comparison of up- and downwash distributions for rolled-up vortex structure as predicted by second moment and by energy formulations assuming vortex sheet composed of point vortices; cosine spacing, $N = 51$; ——, second moment formulation; - - - - - -, energy formulation.

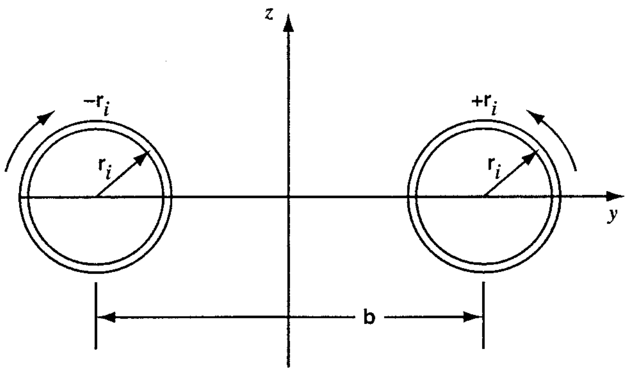


Fig. 9 Vortex ring approximation to rolled-up vortex pair used in attempt to derive energy relationship.

loadings, it is found that the energy formulation is more cumbersome to use and that it sometimes provides erratic results near the center of the vortex where $r_n \approx 0$. Furthermore, it was found that in some cases it is necessary to use far more point vortices in the analysis to get reliable results. It is concluded, therefore, that a formulation based on the second moment of circulation is more robust, more reliable, and easier to use than a formulation based on energy. It is interesting to note, however, that the increments in energy are approximately but not exactly conserved during the roll-up process. In an approximate method, such as Betz', the agreement is quite good considering the number of assumptions made in the theory.

The availability of a fairly simple relationship for the energy increment in the vortex sheet and its corresponding form in the vortex ring located in the rolled-up vortex [Eq. (18)], suggests that similar simple relationships might be found for other characteristics of the roll-up process. It is pointed out that Eq. (18) was readily found because the analysis was restricted to one side of the wake so that the vortex streamlines could be assumed to be concentric circles. If the extended-Betz methods are to be improved, any number of areas for analysis using energy are available. To illustrate the procedures to be tried, the concentric ring geometry used to derive Eqs. (14–18) is modified to analyze the redistribution of vorticity of a pair of point vortices in the vortex sheet into a vortex ring on each side of the wake centerline (Fig. 9). Such a solution would be a first step in an extension of the original Betz method to include both sides of the wake. Unfortunately, it was found that the integrations required could not be carried out analytically even for such a modest extension. The complexity arises because the streamlines are not concentric circles. This exercise again illustrates that the improved guidelines for extended-Betz methods[1] are difficult to derive and that the results may be too complex for intuitive guidance or practical analysis of wakes.

Concluding Remarks

Although extended-Betz methods provide reasonable results for and insight into the roll-up process for complex lift-generated vortex sheets, a need does exist to improve the reliability of the method. The study described here indicates that the improved guidelines will not be provided by the invariants for the time-dependent motion of two-dimensional vortices. The improved guidelines must then either be sought elsewhere or the rolled-up structure of vortex wakes must be determined by other techniques, e.g., numerical analysis. The present study did again find that the complexity and duration of the roll-up process for wakes with multiple vortex pairs make it difficult to develop a simple set of guidelines for the division and roll-up of vortex sheets. Because the objective of the study was not achieved, and both the basic- and extended-Betz methods are simple, easy to use, and do provide an estimate of the far-field vortex structure, it is recommended that the extended-Betz methods continue to be used. It should always be recognized, however, that the predictions are based on a number of approximations that restricts their use to certain experimental and theoretical applications.

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